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Schwarzschild interior solution and the truncated Maxwell fish-eye

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Abstract. This note is based on the observation that the geometric optical behaviour of an object S described by the Schwarzschild interior solution is formally exactly that of the Maxwell fish-eye, truncated at some finite radius. Since the explicit point characteristic of the fish-eye and the character and disposition of rays within it may be obtained without having to solve any ray equations, the imagery of S is fully known. Of particular interest are the conditions under which a point source I in S has a real image I′ in S, granted that one considers such an image to exist if at least some rays from I mutually intersect in I′.

1. Introduction

The Schwarzschild interior solution g_i, that is, the metric of a region of space–time filled with a static, spherically symmetric distribution of fluid S of constant density ρ, has been derived and discussed a great many times despite the unphysical nature of S—because of the constancy of ρ it is acausal. Presumably the attention bestowed upon S is a result of the ease with which the explicit form of g_i may be found, whilst its unphysical character becomes less significant when it is regarded as a limiting case of the class of regular spheres whose density does not increase outwards (Buchdahl 1959). At any rate, granted the heuristic prominence of S, it seems to be appropriate to investigate any of its properties which appear not to have been described before, in particular its optical properties. The question is, how does light—or physically perhaps a little less unrealistically, how do neutrinos—propagate within S on the level of geometrical optics? To answer it, one might integrate the equations for the null geodesics; but it is far simpler here to determine the point characteristic V. To this end it is of advantage to make use of the conformal flatness of g_i. If isotropic coordinates are chosen, one is at once led to the conclusion that the optics of S is formally exactly that of the Maxwell fish-eye, truncated at some finite radius. The optical point characteristic of S is therefore known.

2. The refractive index

When isotropic coordinates are chosen, the generic form of the metric is

$$ds^2 = -q^2(r)(dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2) + f^2(r)\, dt^2 \tag{2.1}$$

and the conformal flatness of this metric, i.e. the vanishing of the Weyl tensor, is assured if and only if

$$f = (A + Br^2)q, \tag{2.2}$$

where A and B are constants. This shows immediately that (when the coordinates are isotropic) the generic optical behaviour of S is formally that of a classical dielectric medium whose refractive index is (Buchdahl 1970)

$$N = q/f = (A + Br^2)^{-1} \tag{2.3}$$

which is just that of the Maxwell fish-eye.

Let

$$M^* := (4\pi\rho/3)R^3, \tag{2.4}$$

where R is the (isotropic coordinate) radius of S. M^* is not the (active) mass M of S. In fact, if

$$\chi := M/2R, \qquad \chi^* = M^*/2R, \tag{2.5}$$

then (Kramer and Neugebauer 1971) χ is a real root of the sextic equation

$$\chi = \chi^*(1+\chi)^6 \tag{2.6}$$

and

$$q = \frac{(1+\chi)^3}{1+\chi r^2/R^2}, \qquad f = \frac{(1-2\chi)+\chi(2-\chi)r^2/R^2}{(1+\chi)(1+\chi r^2/R^2)}. \tag{2.7}$$

A standard form of the refractive index function of the fish-eye is (Born and Wolf 1959)

$$N = N_0/(1 + r^2/a^2), \tag{2.8}$$

where N_0 and a are constants. Comparison with (2.3) shows that here one has

$$a^2 = (1-2\chi)R^2/(2-\chi). \tag{2.9}$$

Also $N_0 = (1+\chi)^4/(1-2\chi)$, which is acceptable since the finiteness of the central pressure p_c requires that

$$\chi < \tfrac{1}{2}. \tag{2.10}$$

(Apart from this the actual value of N_0 is immaterial.) It may be noted that, since $p_c/\rho = \chi/(1-2\chi)$, the limitation $3p/\rho \leq \beta$, where β is an assigned positive constant, implies the restriction

$$\chi \leq \beta/(3+2\beta). \tag{2.11}$$

Usually one takes $\beta = 1$ or, more rarely, $\beta = 3$. For these (2.11) gives

$$\chi_{\beta=1} \leq \tfrac{1}{5}, \qquad \chi_{\beta=3} \leq \tfrac{1}{3}. \tag{2.12}$$

3. Existence of pairs of conjugate points

In a (complete) Maxwell fish-eye every ray is a circle. All rays which originate from a point $P[r = r_1]$, $r_1 < a$, pass through a point $P'[r = r_1']$, where $r_1 r_1' = a^2$. In S, however, the effects of truncation must be taken into account.

To begin with, a given point $P[r_1]$ – of course $r_1 < R$—will have a real image $P'[r_1']$ only if $r_1' < R$ also; and it is taken for granted that not all rays from P need pass through P'. The two conditions $r_1 < R$, $r_1' < R$ together require that $r_1' r_1 (= a^2) < R^2$. In view of (2.9), P will thus have a real image only if $\chi^2 - 4\chi + 1 < 0$, or

$$\chi > \chi_0 := 2 - \sqrt{3} \approx 0.268. \tag{3.1}$$

In other words, when $\chi < \chi_0$ no point P in S has a conjugate image point P' in S, that is, no pair of rays through P mutually intersect anywhere else in S. (3.1) is in conflict with the first of the inequalities (2.12), though not with the second.

By suitably choosing the unit of length one can arrange a to have the value unity; and this will henceforth be taken to have been done.

4. The disposition of rays

The explicit form of the optical point characteristic V of S can be obtained in ways which circumvent the cumbersome integration of the equations it satisfies. In fact (Buchdahl 1972, 1975),

$$V = \sin^{-1} \tau, \tag{4.1}$$

where

$$\tau^2 = (\xi - 2\eta + \zeta)/(1+\xi)(1+\zeta), \tag{4.2}$$

in terms of the rotational invariants

$$\xi := x'^2 + y'^2 + z'^2, \qquad \eta := x'x + y'y + z'z, \qquad \zeta := x^2 + y^2 + z^2. \tag{4.3}$$

Here $Q[x, y, z]$ and $Q[x', y', z']$ are two points on an arbitrary ray C, granted that the coordinates are now so chosen that in (2.1) the factor multiplying $-q^2$ becomes $dx^2 + dy^2 + dz^2$. If $\boldsymbol{e} \equiv (\alpha, \beta, \gamma)$, $\boldsymbol{e}' \equiv (\alpha', \beta', \gamma')$ are the usual tangents to C at Q and Q', respectively, one has in particular

$$N\boldsymbol{e} = -\operatorname{grad} V = \frac{\tau}{(1-\tau^2)^{1/2}} \left(\frac{\boldsymbol{r}' - \boldsymbol{r}}{\xi - 2\eta + \zeta} + \frac{\boldsymbol{r}}{1+\zeta} \right). \tag{4.4}$$

Now, without loss of generality one may take C to lie in the xy plane: $z = z' = \gamma = \gamma' = 0$. Take Q to be the fixed initial point $P[-r, 0, 0]$. Then $\beta/\alpha =: \tan\omega$ is the initial slope of the ray through P and Q', i.e. ω is the angle the ray makes with the x axis at P. α and β, as functions of x' and y', are read off from (4.4) and so one obtains immediately the equation of C:

$$[x' - (1-r^2)/2r]^2 + [y' + (1+r^2)/2r \tan\omega]^2 = [(1+r^2)/2r \sin\omega]^2. \tag{4.5}$$

Thus C is a 'circle' of radius

$$\mathscr{R} := (1+r^2)/2r\, |\sin\omega|, \tag{4.6}$$

with centre at $[(1-r^2)/2r, -(1+r^2)/2r \tan\omega] =: (u, v)$, say. Equation (4.5) is, of course, satisfied when $x' = 1/r$, $y' = 0$, independently of the value of ω.

To find the points of intersection of a ray with the boundary of S, set $x' = R\cos\psi$, $y' = R\sin\psi$ in (4.5). The equation for $\sin\psi$ is then

$$(u^2 + v^2)\sin^2\psi - 2vK \sin\psi + (K^2 - u^2) = 0, \tag{4.7}$$

where $K := (1-R^2)/2R$. Real roots exist only if $u^2+v^2 \geqslant K^2$, or

$$|\sin\omega| \leqslant R(1+r^2)/r(1+R^2) =: \sigma, \tag{4.8}$$

say. It evidently suffices to take $|\omega| \leqslant \pi/2$, since the angles ω and $\pi-\omega$ belong to one and the same circular arc. If a^2 is restored explicitly in (4.8) and then eliminated by means of (2.9), one has

$$\sigma = \frac{(1-2\chi)+\chi(2-\chi)(r/R)^2}{(1-\chi^2)(r/R)}. \tag{4.9}$$

Here the condition $\chi > \chi_0$ should re-emerge from the condition that *some* rays through P be complete 'circles' or, in other words, that the condition $|\sin\omega| > \sigma$ can be satisfied. This will be the case provided that $\sigma < 1$. This inequality can be violated when the value of r is sufficiently close to a unless $\chi < \chi_0$, as expected.

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